A Discontinuous Petrov-Galerkin method for the transport equation

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Numerically solving the nonlinear system in our porous media model, namely the continuity and the Darcy-Forchheimer equation, requires an iterative process which involves, in one way or the other, a linearization of the system and thus gives rise to systems of linear transport equations. For such an approach to be successful at least the following two issues need to be addressed: first, to cover a realistic range of velocities Galerkin type discretizations need to be stabilized. In absence of any viscosity a proper choice of stabilization parameters, e.g., in the most prominent stabilization method, the SUPG scheme, is delicate. Second, conventional schemes do not come with an a-posteriori error control which is needed to monitor the convergence.

Therefore, as an alternative central tool, we present here a recently developed Discontinuous Petrov-Galerkin (DPG) formulation of transport equations which in the linear case offers, in contrast to conventional schemes, the following features: the scheme is stable in a very strong sense which, in particular entails that errors are uniformly bounded from below and above by (computable) residuals in a suitable dual norm. This is the basis for a rigorous a-posteriori error control. Moreover, the resulting approximate solutions realize up to a uniform constant factor the best possible accuracy obtainable from the underlying trial space. In this report we summarize and explain the underlying DPG stability concepts for the linear case and conclude by explaining how to exploit them for the treatment of the nonlinear system.

1. Introduction

Transpiration cooling is a promising concept for the reduction of thermal loads at walls in future space transportation systems. It is based on the injection of a coolant into a boundary layer through a porous material. The injection is driven by the pressure difference between the hot gas flow and the coolant reservoir. The hot gas in the boundary layer is displaced and a protective layer with lower temperature is formed on the surface of the cooled structure. An accompanying effect is the significant reduction of the resulting wall heat flux and the skin friction [1].

In recent work [1–6] transpiration cooling in a subsonic turbulent channel has been investigated numerically using a two-domain approach. The turbulent hot gas channel flow is modeled by the compressible Reynolds-averaged Navier-Stokes equations whereas the porous medium flow is modeled by the continuity equation, the Darcy-Forchheimer equation and two temperature equations for both fluid and solid material. These models are coupled by interface conditions.

In the porous medium flow the continuity equation and the Darcy-Forchheimer equation
form a system of non-linear hyperbolic equations. So far the numerical solution of this system is based on an SUPG stabilized Finite Element Galerkin method. In general, plain Galerkin schemes [7] applied to pure transport or convection-dominated problems suffer from stability problems and therefore have to be complemented by stabilization concepts.

Standard approaches are to add artificial viscosity, cf. [8, 9], or using upwind discretization, cf. [10]. Such stabilization techniques are applied on the discrete level and typically lower the consistency order of the discretization. Moreover, one needs to choose stabilization parameters which, for instance, for the perhaps most prominent variant, SUPG (Streamline Upwind Petrov-Galerkin), the choice of such stabilization parameters is well understood only for linear convection-diffusion problems so that, in general, the numerical solution may still exhibit unphysical oscillations or oversmearing. More importantly perhaps even the stabilized variational formulations do not allow one to tightly estimate errors by residuals, i.e., errors (in a given norm) cannot be bounded from below and above by residuals (in a suitable dual norm). This latter property, however, is essential for deriving rigorous tight a posteriori error bounds which in turn are particularly desirable for coupled problems where the approximate solution to one component is used as input when updating the solution in the other component.

The deficiencies mentioned above will be aggravated when entering desirable flow regimes with higher pressure differences and larger velocities. In order to be able to treat such regimes with reliable numerical tools a major work package during the past year has been the exploration and development of alternative numerical approaches. This concerns work on theoretical foundations as well as related implementations, see [11–13]. The basic strategy has been to prepare a complete theoretical foundation for the linear case, i.e., for linear transport equations with variable convection fields and to use these results then for the non-linear systems through contriving suitable linearizing iterations in function space [12]. In this report we outline the corresponding methodological developments and the first steps of their realization.

Recently, conceptually different alternatives have been proposed that aim at stabilizing the problem already on the continuous infinite dimensional level. This is done by contriving variational formulations for the original continuous problem that are well-conditioned in a sense to be made precise below. This spares one tuning stabilization parameters and leads to tight residual based a posteriori error bounds. The key is that to arrive at such well-conditioned weak formulations one may have to choose the (infinite-dimensional) trial space that hosts the exact solution different from the test-space that provides the weak conditions for the solution. Thus, rather than a Galerkin framework one deals with a Petrov-Galerkin framework, already on the infinite dimensional level. The price to be paid for the entailed excellent stability of the formulation is to identify for the finite-dimensional discrete problems suitable finite-dimensional test spaces that inherit the stability of the infinite-dimensional problem, see e.g. [14–16] for the basic underlying idea. It turns out, however, that finding such optimal test spaces is in general as hard as solving the problem in the first place. Several strategies have therefore been developed to construct near-optimal test spaces that can be computed at affordable cost. One such paradigm, and this is the one considered in the present article as well, is the so-called Discontinuous Petrov Galerkin (DPG) framework that has been mainly advocated by Demkowicz and Gopalakrishnan [17–19] and further advanced e.g. in [11,20–22] and in further works cited in those papers. The main reason for considering Discontinuous Galerkin formulations already for the infinite dimensional problem is
that this offers the chance of finding near-optimal localized test spaces yielding a well-conditioned variational formulation of the original problem with the desirable properties mentioned above. To obtain from this infinite-dimensional formulation a fully discrete finite-dimensional problem, the optimal test space for a given finite-dimensional trial space would ideally be obtained with the aid of localized projections from an infinite-dimensional test-search space. Thus for a computable variant one needs to identify suitable (localized) finite-dimensional test search spaces. Here one faces two conflicting objectives: on the one hand, these finite-dimensional spaces need to be large enough to warrant stability; on the other hand, they should be kept as small as possible to keep the computational work acceptable. This issue has to be addressed for each individual problem separately and we shall briefly comment on this for the case of linear transport.

DPG concepts as they stand are formulated for linear problems while our scenario of interest concerns a nonlinear hyperbolic system of transport equations. Our envisaged approach is to linearize this system in the function space, e.g., in the context of a Newton or Picard iteration, and then solve each linear subproblem - a system of linear transport equations - with a DPG method with a posteriori controlled error tolerances. In order to explain the relevant DPG concepts, as a guideline, we summarize basic ingredients and illustrate them for a simple linear scalar transport equation following [11]. This example is not only a natural first step towards the actual target application but lacks in some sense all the favorable features of elliptic boundary value problems for which variational methods are in general by far best understood.

In Section 2 we introduce the Petrov-Galerkin method. We discuss the connection of the trial and the test space and how optimal test functions help to gain control over the condition number of the underlying variational problem. We present the concept of broken test spaces that lead to localized test-search spaces thereby reducing computational costs. In Section 3 we apply this method to the transport equation and provide first numerical results. We present the theoretical DPG setup for the transport part of the porous media problem in Section 4. We conclude with a summary and a brief outlook in Section 5.

2. Variational Formulations

2.1. A Guiding Example

To illustrate how a Petrov-Galerkin discretization with (near-)optimal test functions works, we consider as a guiding example the following simple linear transport equation

\[ p \cdot \nabla u + cu = f_0 \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \Gamma_- \]

(2.1)

where

\[ \Gamma_{\pm} := \{ x \in \partial \Omega \mid \pm p(x) \cdot n(x) < 0 \} \]

Thus, the inflow boundary \( \Gamma_- \) is the portion of the boundary \( \partial \Omega \) on which the convection direction points into the domain. Keeping our envisaged application in mind, we allow the convection field \( p \) to depend on the spatial variable \( x \) but assume throughout some mild regularity such as essentially bounded divergence of \( p \). Solutions to such equations may exhibit discontinuities along layers when the right-hand side has jumps or when one imposes non-trivial discontinuous inflow boundary conditions. Therefore, it is useful to consider weak formulations of (2.1) for which derivatives are no longer required to have
a pointwise meaning. Formally, such a weak formulation is obtained by multiplying (2.1) with a test function and integrating over the computational domain. The problem then becomes: given \( f_0 \), find \( u \) such that

\[
\int_{\Omega} (p \cdot \nabla u + cu) v \, dx = \int_{\Omega} f_0 v \, dx, \quad u|_{\Gamma_-} = 0,
\]

holds for all test functions \( v \in L^2(\Omega) \). For this to be meaningful one only requires that together with \( u \) itself its directional derivatives \( p(x) \cdot \nabla u(x) \) are (Lebesgue) square integrable. Thus, writing as usual \( ||v||^2_{L^2(\Omega)} := \int_{\Omega} |v(x)|^2 \, dx \), the solution is to belong to the space

\[
H(p; \Omega) := \{ u \in L^2(\Omega) : p \cdot \nabla u \in L^2(\Omega) \}
\]

endowed with the norm

\[
||v||^2_{H(p; \Omega)} := ||v||^2_{L^2(\Omega)} + ||p \cdot \nabla v||^2_{L^2(\Omega)}.
\]

More precisely, homogeneous inflow boundary conditions are essential and thus need to be incorporated in the trial space. The corresponding closed subspace of \( H(p; \Omega) \) is denoted by \( H_{0, \Gamma_-}(p; \Omega) \) and defined as

\[
H_{0, \Gamma_-}(p; \Omega) := \text{clos}_{H(p; \Omega)} \{ u \in C^1(\Omega) : u = 0 \text{ on } \Gamma_- \}.
\]

In summary, for the variational formulation (2.2) of the transport equation (2.1) we have employed the pair of trial and test spaces

\[
U = H_{0, \Gamma_-}(p; \Omega), \quad V = L^2(\Omega),
\]

Obviously \( U \neq V \) and the trial space \( U \) depends explicitly on the convection field \( p \).

This latter fact is a drawback. In fact, the elements of \( H(p; \Omega) \) are allowed to have discontinuities as long as they are tangential to a streamline. Thus, the trial space \( H_{0, \Gamma_-}(p; \Omega) \) depends delicately on the convection field \( p \); one has \( H(p; \Omega) \neq H(p; \Omega) \) whenever \( \tilde{p} \neq p \) is any (non-parallel) perturbation of \( p \). Measuring errors in the natural graph-norm \( \| \cdot \|_{H(p; \Omega)} \) is particularly undesirable when linear transport equations arise as iterates in the solution of a nonlinear problem where the convection field depends on the preceding iterate and hence varies. This can be avoided by further weakening regularity requirements on \( u \) by employing a different variational formulation. Applying integration by parts to the left-hand side of (2.2), one obtains

\[
\int_{\Omega} (uc - \text{div}(v p)) \, dx = \int_{\Omega} u ((c - \text{div}p)v - p \cdot \nabla v) \, dx = -\int_{\Omega} n \cdot puvd + \int_{\Omega} f_0 vd, \quad (2.7)
\]

where we assume for the moment that the boundary integral exists. To that end, note first that the left-hand side of (2.7) is well-defined when the solution \( u \) is only in \( L^2(\Omega) \) provided that \( ((c - \text{div}p)v - p \cdot \nabla v) \in L^2(\Omega) \). When \( c, \text{div}p \) are essentially bounded this is indeed the case the test function \( v \) belongs to \( H(p; \Omega) \). Moreover, since we impose conditions on \( u \) on \( \Gamma_- \) it is natural to constrain the test functions on the outflow boundary \( \Gamma_+ \). Specifically, when \( v \in H_{0, \Gamma_+}(p; \Omega) \) we know from corresponding trace theorems that \( v|_{\Gamma_+} \in L^2(\Gamma_+; \n \cdot p) \). Thus, when \( u \) vanishes on \( \Gamma_- \) and \( v \) on \( \Gamma_+ \), the boundary integral on the right-hand side of (2.7) vanishes. More generally, we are allowed to impose an inflow boundary condition \( g \) on \( u \) provided that \( g \in L^2(\Gamma_-; \n \cdot p) \), where

\[
||g||^2_{L^2(\Gamma_-)} = \int_{\Gamma_-} |\n \cdot p| g|^2 \, ds.
\]
The variational problem then becomes
\[ \int_{\Omega} u((c - \text{div} p)v - p \cdot \nabla v)\,dx = \int_{\Gamma} |n \cdot p|\,g\,ds + \int_{\Omega} f_0\,v\,dx, \quad v \in H_{0, \Gamma^+}(p; \Omega). \] (2.8)

Note that now the boundary conditions are part of the variational formulation and thus have become natural ones. In summary, (2.8) is an alternate variational formulation with trial and test spaces
\[ U = L_2(\Omega), \quad V = H_{0, \Gamma^+}(p; \Omega). \] (2.9)

In all cases the trial and test spaces are easily seen to be Hilbert spaces, i.e., they are complete and their norms are induced by an inner product. Of course, one needs to assert that the problems (2.2) or (2.8) have unique solutions in the respective trial spaces and that these solutions have the stability properties discussed in the introduction. The relevant concepts are summarized in the next section.

2.2. The condition of a problem and inf-sup stability

The left-hand sides of (2.2) and (2.8) both define bilinear forms, namely
\[ b(u, v) = \int_{\Omega} (p \cdot \nabla u + cu)\,v\,dx \] for (2.2) and\[ b(u, v) = \int_{\Omega} ((c - \text{div} p)v - p \cdot \nabla v)\,dx \] for (2.8), acting on \( U \times V \) with the respective choices of \( U \) and \( V \) from (2.6) and (2.9). In the variational formulations the right-hand side data play the role of functionals acting on the test functions, i.e., when for instance \( f_0 \in L_2(\Omega) \) the right-hand side induces a functional on the test space \( V \) by
\[ f(v) := \int_{\Gamma} |n \cdot p|\,g\,ds + \int_{\Omega} f_0\,v\,dx \]
for (2.2) and (2.8), respectively. One can show that in both cases one has \( |f(v)| \leq C \|v\|_V \) holds for some constant independent of \( v \). Hence, \( f : V \to \mathbb{R} \) is a bounded linear functional and thus an element of the normed dual \( V' \) of \( V \) endowed with the norm
\[ \|f\|_{V'} = \sup_{v \in V} \frac{f(v)}{\|v\|_V}. \] (2.10)

Thus, the common abstract form of (2.2) and (2.8) reads as follows: given a bilinear form \( b(\cdot, \cdot) : U \times V \to \mathbb{R} \) on a pair of Hilbert spaces \( U, V \), and linear functional \( f \in V' \), find \( u \in U \) such that
\[ b(u, v) = f(v) \quad \forall v \in V \] (2.11)
holds. At this point the main distinction from standard weak formulations of elliptic problems is that we allow the trial space \( U \) to be different from the test space \( V \).

We characterize next the unique solvability of (2.11). To that end, note that in both examples (2.2) and (2.8) the norms are chosen so that, as a simple consequence of the Cauchy Schwarz inequality,
\[ |b(u, v)| \leq C_b \|u\|_U \|v\|_V, \quad u \in U, \ v \in V. \] (2.12)

In fact, in the case (2.8) one obtains
\[ \left| \int_{\Omega} u(cv - \text{div}(pv))\,dx \right| \leq \|u\|_{L_2(\Omega)} \|cv - \text{div}(pv)\|_{L_2(\Omega)} \]
\[ \leq (1 + (\|c\|_{L_\infty(\Omega)} + \|\text{div} p\|_{L_\infty(\Omega)})^2)^{1/2} \|u\|_{L_2(\Omega)} \|v\|_{H(p; \Omega)}, \] (2.13)
i.e., for $U = L^2(\Omega), V = H_0, \Gamma_\eta (p; \Omega)$ one has $C_b \leq (1 + (\|c\|_{L^\infty(\Omega)} + \|\text{div} p\|_{L^\infty(\Omega)})^2)^{1/2}$.

To properly interpret the boundedness of the bilinear form $b(\cdot, \cdot)$ it is convenient to reinterpret (2.11) as an operator equation. In fact, since the quantity
\[ \|B\| := \sup_{0 \neq u \in U, 0 \neq v \in V} \frac{b(u, v)}{\|u\|_U \cdot \|v\|_V} \leq C_b \tag{2.14} \]
the Riesz representation theorem asserts that
\[ (Bu)(v) := b(u, v), \quad u \in U, v \in V, \tag{2.15} \]
defines a bounded linear operator from the trial space $U$ to the dual of the test space $V'$. In fact, denoting by $\mathcal{L}(U, V')$ the space of bounded linear operators from $U$ to $V'$, we have by definition $\|B\|_{\mathcal{L}(U, V')} = \sup_{u \in U} \|Bu\|_{V'} = \|B\|$ where we have used (2.10) and the definition (2.14) of $\|B\|$ in the last step.

Therefore, (2.11) is equivalent to the operator equation: given $f \in V'$, find $u \in U$ such that
\[ Bu = f. \tag{2.16} \]
In both cases (2.2) and (2.8) $B$ represents the transport operator $u \mapsto p \cdot \nabla u + cu$ but as a mapping between different pairs of spaces $U, V'$.

Unique solvability of (2.11) is then equivalent to bijectivity of $B : U \to V'$ while stability means that $B^{-1}$ is also bounded, i.e., $B^{-1} \in \mathcal{L}(V', U)$. The Banach-Necas Theorem [7] states that $B$ is boundedly invertible, i.e., $B \in \mathcal{L}(U, V')$, $B^{-1} \in \mathcal{L}(V', U)$ if and only if (2.12) as well as the inf-sup conditions
\[ \sup_{0 \neq u \in U} \inf_{0 \neq v \in V} \frac{b(u, v)}{\|u\|_U \cdot \|v\|_V} \geq \gamma \quad \text{and} \quad \inf_{0 \neq u \in U} \sup_{0 \neq v \in V} \frac{b(u, v)}{\|u\|_U \cdot \|v\|_V} \geq \gamma. \tag{2.17} \]
are valid for some $\gamma > 0$. The second part just says that $B : U \to V'$ is injective while the first part implies injectivity of the adjoint operator $B^* : V' \to U$ defined by
\[ (B^* v)(u) = (Bu)(v), \tag{2.18} \]
and thus bijectivity of $B : U \to V'$ together with the bound
\[ \|w\|_U \leq \frac{1}{\gamma} \|Bw\|_{V'}, \quad w \in U. \tag{2.19} \]
In particular, this means that the solution $u$ to (2.11) is bounded by the data $\|u\|_U \leq \gamma^{-1} \|f\|_{V'}$, which is commonly referred to as stability. Obviously, the larger $\gamma$ the better the stability. More precisely, in analogy to linear systems of equations we can associate with the variational formulation (2.11) or equivalently with the operator equation (2.16), the condition number
\[ \kappa_{U, V'}(B) := \|B\|_{\mathcal{L}(U, V')} \cdot \|B^{-1}\|_{\mathcal{L}(V', U)} \leq \frac{C_b}{\gamma}. \tag{2.20} \]
The smaller $\kappa_{U, V'}(B)$ the better conditioned is the variational formulation (2.11). The relevance of a small condition number (or a good upper bound) is explained by the relations
\[ C_b^{-1} \|Bw - f\|_{V'} \leq \|w - u\|_U \leq \gamma^{-1} \|Bw - f\|_{V'}, \quad w \in U, \tag{2.21} \]
which follows from $Bw - f = B(w - u)$, (2.12) and (2.19). Thus, errors measured in the norm $\|\cdot\|_U$ are bounded from above and below by the corresponding residual measured in the dual norm $\|\cdot\|_{V'}$. Clearly, the smaller $\kappa_{U, V'}(B)$ the tighter are these bounds. The
importance of (2.21) lies in the following fact. Since \( u \) is unknown we cannot estimate its deviation from an approximation \( w \) directly. However, since the residual \( Bw - f \) contains for a given approximation only known quantities it can, at least in principle, be evaluated. This is the most important starting point for deriving rigorous \textit{a posteriori} error estimates which requires dealing with the dual norm that measures the residual.

It is instructive to discuss what this means for the variational problem (2.8). Note first that the adjoint \( B^* \) of \( B \) is just \( v \mapsto (c - \text{div} p)v - p \cdot \nabla v \). Under mild assumptions on \( p, c \) in (2.1) one can show that \( B^* \) and \( B \) are injective on dense subsets of \( H_0,Γ_+ (p; Ω) \), \( L_2(Ω) \) respectively, see e.g. [15]. Thus, recalling that in this case \( U = L_2(Ω) = U' \)

\[
\|e\|_{V'} := \|B^*v\|_{U'} = \|cv - \text{div}(pv)\|_{L_2(Ω)}
\]

(2.22) is a norm on \( V = H_0,Γ_+ (p; Ω) \). Employing this norm in place of \( \| \cdot \|_{H(p; Ω)} \), the first line in (2.13) says that then

\[
\inf_{0 \neq u \in U} \sup_{0 \neq v \in V} b(u, v) \|u\|_{U'} \|v\|_V = \inf_{0 \neq u \in U} \frac{b(u, (B^*)^{-1}u)}{\|u\|_{U'} \|(B^*)^{-1}u\|_V} = \frac{\|B^*v\|_V}{\|Bv\|_V} = 1,
\]

(2.24)
i.e., one has in this case \( C_b = 1 \). Moreover, given \( u \in U \), taking \( v = (B^*)^{-1}u \in V = H_0,Γ_+ (p; Ω) \) yields

\[
\inf_{0 \neq u \in U} \sup_{0 \neq v \in V} b(u, v) \|u\|_{U'} \|v\|_V = \inf_{0 \neq u \in U} \frac{b(u, (B^*)^{-1}u)}{\|u\|_{U'} \|(B^*)^{-1}u\|_V} = 1,
\]

(2.24)

because \( \|(B^*)^{-1}u\|_V = \|u\|_{U'} = \|u\|_{L_2(Ω)} \) and \( b(u, (B^*)^{-1}u) = (Bu)((B^*)^{-1}u) = \|u\|_{Ω}^2 = \|u\|_{L_2(Ω)}^2 \). The first inf-sup condition in (2.17) can be treated similarly. Given \( v \in V \), we simply take \( u = B^*v \) to conclude that \( b(B^*v, v) = (B^*v)(B^*v) = \|B^*v\|_{L_2(Ω)}^2 \) to continue arguing then as before.

Hence, for this pair of norms \( \| \cdot \|_U, \| \cdot \|_V \) one has, in view of (2.23), even

\[
\kappa_{U, V'}(B) = 1,
\]

(2.25)
i.e., the variational problem is \textit{perfectly} conditioned. Moreover, one can show under the above assumptions on \( p, c \) (see [15]) that the original norm \( \| \cdot \|_V = \| \cdot \|_{H(p; Ω)} \) and the "ideal" norm \( \| \cdot \|_{V'} \) from (2.22) are equivalent with constants depending only on the parameters \( p, c \). Hence, one has also in this case a positive inf-sup constant \( \gamma \) which is safely bounded away from zero so that \( \kappa_{U, V'}(B) \) is of moderate size.

In summary, one sees that, although one cannot resort to any of familiar features of elliptic theory, one obtains well-conditioned variational formulations for transport equations with entailed sharp \textit{a posteriori} residual based error bounds. For analogous results concerning convection-diffusion equations with arbitrarily small diffusion see [16].

We emphasize that the discussion so far concerns the condition of the \textit{infinite dimensional} problem. Of course, the rationale is that one cannot expect a discretization to perform well if it is based on a poorly or ill-conditioned infinite-dimensional problem.

2.3. \textit{Optimal and near-optimal test space}

We assume now that the variational problem (2.11) is well-conditioned, i.e., (2.12), (2.17), and hence (2.20) hold. For a given (finite dimensional) trial space \( U^h \subset U \), we would like to find a test space that inherits the stability (2.17) of the infinite dimensional formulation over \( U \) and \( V \). Thus the test space should be contained in \( V \) and will therefore have to be in general different from \( U^h \). A convenient tool for identifying a test space
that is in some sense \textit{optimal} is the so called trial-to-test-operator \(T \in \mathcal{L}(U, V)\) defined by
\[
(Tu,v)_V = b(u,v), \quad u \in U, \ v \in V. \tag{2.26}
\]
In fact, \(T\) is even an isomorphism. To see this we introduce the \textit{Riesz map} \(R_V \in \mathcal{L}(V, V')\) defined by
\[
(R_Vv)(w) = (v, w)_V, \quad \forall \ v, w \in V, \tag{2.27}
\]
where \((\cdot, \cdot)_V\) denotes the inner product in \(V\). \(R_V\) is easily seen to be an isometric isomorphism, i.e.,
\[
\|R_Vv\|_V = \|v\|_V.
\]
We can then write \(b(u,v) = (Bu)(v) = (R_V^{-1}Bu, v)_V\) to conclude that \(T = R_V^{-1}B = R_VB\) and thus the composition of two isomorphisms. Its relevance lies in the fact that for any given \(u \in U\) the element \(Tu \in V\) is its \textit{supremizer} in the sense that
\[
\|Tu\|_V = \sup_{v \in V} \frac{(Tu,v)_V}{\|v\|_V} = \sup_{v \in V} \frac{b(u,v)}{\|v\|_V} = \|Bu\|_V. \tag{2.28}
\]
Thus, for a given \(U^h \subset U\) the space
\[
T(U^h) = \{Tu^h : u^h \in U^h\} \subset V, \tag{2.29}
\]
has the same dimension as \(U^h\) and contains for each \(u^h \in U^h\) its supremizer which, by definition, maximizes the inf-sup constant. Hence
\[
\inf_{u^h \in U^h} \sup_{v \in T(U^h)} \frac{b(u^h,v)}{\|u^h\|_V \|v\|_V} = \inf_{u \in U^h} \sup_{v \in V} \frac{b(u,v)}{\|u\|_V \|v\|_V} \geq \gamma > 0, \tag{2.30}
\]
where we have used (2.17) in the last step. Since the operator \(B_h\) induced by \(b(u_h,v_h)\) over \(U^h \times T(U^h)\) is finite dimensional, injectivity is equivalent to surjectivity. It follows that the finite-dimensional analog to (2.11) - Petrov-Galerkin problem: find \(u^h \in U^h\) such that
\[
b(u^h, v^h) = f(v^h), \quad \forall v^h \in T(U^h), \tag{2.31}
\]
is at least as well-conditioned as the original infinite-dimensional problem (2.11), i.e.,
\[
\kappa_{U^h,T(U^h)}(B_h) \leq \kappa_{U,V}(B). \tag{2.30}
\]
Therefore, the test space \(T(U^h)\) is called \textit{optimal}.

However, to actually determine \(T(U^h)\) one would have to compute
\[
R_V^{-1}B\phi_j, \quad j = 1, \ldots, N_h,
\]
when \(\phi_j, j = 1, \ldots, N_h\), is a basis for \(U^h\). Hence for each \(j\) one has to solve the infinite-dimensional variational (elliptic) problem
\[
(\psi_j, v)_V = b(\phi_j, v), \quad v \in V. \tag{2.32}
\]
This is of very questionable value for two reasons:
(i) each of the problems is still infinite-dimensional and hence has to be discretized in a way that is computationally affordable but preserves stability.
(ii) one has to solve \(\dim(U^h)\) \textit{global} variational problems which are possibly as complex as the discretized problem itself. Moreover, the resulting test basis functions could in general be \textit{global} so that the linear system corresponding to (2.31) has a densely populated matrix.
A natural strategy to address issue (i) is to replace the full space $V$ in (2.32) by a finite-dimensional subspace $V^h \subset V$ which is sometimes called test-search space. In fact, this means to replace the mapping $T$ defined in (2.26) by

$$ (T_h u, v_h)_V = b(u, v_h), \quad v_h \in V^h $$

(2.33)
to arrive at the “practical” Petrov-Galerkin problem

$$ b(u^h, v^h) = f(v^h) \quad \forall v^h \in T_h(U^h). $$

(2.34)

Since in general $T_h(U^h) \neq T(U^h)$ the optimal inf-sup stability (2.30) will no longer hold. However, for $V_h$, sufficiently large one expects to realize an inf-sup constant $\tilde{\gamma} < \gamma$ but as close to $\gamma$ as one wishes. In this sense the choice of the test-search space determines the quantitative stability. The larger $\gamma$ in the first place the more leeway one has, of course, for the discrete problem. The concrete choice of $V^h$ depends on the specific problem. Of course, to keep the computational complexity proportional to $\dim(U^h)$ one would try to keep $\dim(V^h)$ proportional to $\dim(U^h)$, uniformly in $h$. At any rate, if one manages to find for each $U^h$ an acceptable test-search space $V^h$ yielding an inf-sup constant $\tilde{\gamma} > 0$ that stays uniformly bounded away from zero with respect to increasingly larger trial spaces $U^h$, the generalized Ceá-Lemma says that

$$ \|u - u^h\|_U \leq C_b \tilde{\gamma} \inf_{w^h \in U^h} \|u - w^h\|_U, $$

(2.35)
i.e., the Petrov-Galerkin projection provides, up to the constant $C_b/\tilde{\gamma}$, the best possible accuracy that can ever be obtained by the trial space $U^h$.

2.4. Localization

We now address issue (ii) from the preceding section. One strategy to keep the Petrov-Galerkin system matrix sparse and to avoid solving $\dim(U^h)$ global problems is based on the concept of localization. It has been introduced by Demkowicz et al. [18] and leads to the so-called Discontinuous Petrov-Galerkin (DPG) formulation. It starts with a mesh-dependent variational formulation for the full infinite dimensional problem, however, the test space $V$ can now be chosen as a product of broken spaces. More precisely, given a mesh $\Omega_h$ partitioning $\Omega$ one takes

$$ V := \prod_{K \in \Omega_h} V_K $$

(2.36)

where each space $V_K$ is endowed with an inner product $(\cdot, \cdot)_{V_K}$ so that

$$ \|v\|_V = \left( \sum_{K \in \Omega_h} \|v_K\|^2_{V_K} \right)^{1/2} $$

(2.37)

which is commonly referred to as a broken norm. As will be exemplified below for the transport equation, the bilinear $b(\cdot, \cdot)$ in (2.11), representing an infinite-dimensional discontinuous Galerkin formulation, can be written as

$$ b(u, v) = \sum_{K \in \Omega_h} b_K(u, v), $$

(2.38)
where the “local” bilinear forms are associated with the cells \( K \in \Omega_h \). As a consequence, the trial-to-test map \( T : U \rightarrow V \) now takes the form

\[
Tu = \sum_{K \in \Omega_h} T_K u
\]

(2.39)

where

\[
(T_K u, v)_{V_K} = b_K(u, v) \quad \forall v \in V_K.
\]

(2.40)

Now the computation of a test basis function becomes a local, yet still infinite-dimensional, problem.

To deal with the infinite-dimensionality one can proceed as indicated before by replacing each infinite-dimensional space \( V_K \) by a finite dimensional test-search space \( V^h_K \subset V_K \). The computation of test functions then reduces to solving problems of the type

\[
(T_K^h u^h, v^h)_{V_K} = b_K(u^h, v^h) \quad \forall v^h \in V^h_K.
\]

(2.41)

When \( u^h \) runs through a basis of a finite-dimensional DG trial space \( U^h \) (spanned by functions supported on single cells and their boundary), due to the locality of \( V^h_K \) one has to solve at most the order \( \dim(U^h) \times \max_{K \in \Omega_h} \dim(V^h_K) \) small dimensional local problems. Thus, the total work scales with \( \dim(U^h) \) if the local test-search spaces \( V^h_K \) have a uniformly bounded finite dimension. The system matrix resulting from (2.34) then remains sparse.

We detail these concepts in the next section for the simple transport equation discussed in Section 2.1.

### 3. Application to the transport equation

#### 3.1. A DPG-variational formulation

We return to the transport equation (2.1). The first step is to derive an infinite-dimensional variational formulation of (2.1) but in a DG context. To describe this, let \( \Omega_h \) be a set of disjoint, open polyhedral domains such that \( \bar{\Omega} = \bigcup_{K \in \Omega_h} \bar{K} \). Here \( h \) is a grid parameter representing, e.g., the mesh size. For each cell \( K \) we denote by \( n_K = n_K(x) \) the unit outward normal at \( x \in \partial K \) (except at points on lower dimensional facets of \( \partial K \) where the normal is not defined). This allows us to determine the inflow, outflow or characteristic boundary portions of \( \partial K \), i.e., \( \partial K = \partial K_+ \cup \partial K_- \cup \partial K_0 \). A vital part of the DG paradigm is the mesh skeleton defined as

\[
\partial \Omega_h := \bigcup_{K \in \Omega_h} \partial K \setminus \partial K_0.
\]

(3.1)

According to the standard DG approach, one applies integration by parts as in (2.7) cellwise to obtain for each \( K \in \partial \Omega \) formally

\[
\int_K (e - \text{div} p) v - p \cdot \nabla v) dx + \int_{\partial K} n \cdot p v d\Gamma, \quad v \in V_K.
\]

(3.2)

Since (in contrast to the classical DG approach) it is a crucial constituent of the DPG concept to formulate the variational problem first in the infinite-dimensional setting one now faces the following principal problem: since derivatives have been completely moved
to the test side - sometimes referred to as ultra-weak formulation - one seeks the solution cellwise in \( L_2(K) \). However, for \( L_2 \) functions, traces, i.e., restrictions to lower dimensional manifolds such as boundaries are generally not defined. Therefore one introduces an additional auxiliary unknown \( \theta \) living on the mesh-skeleton and replaces \( u \) in \\
\[
\int_{\partial K} \mathbf{n} \cdot \mathbf{p} u \, dv \, dx \text{ by } \theta. \]
Then the local DG-bilinear form \( b_K((\cdot, \cdot); v) \) appearing in (2.38) takes the form
\[
b_K((u, \theta); v) = \int_K (c - \text{div} \mathbf{p} v - \mathbf{p} \cdot \nabla v) \, dx + \int_{\partial K} \mathbf{n} \cdot \mathbf{p} \theta \, dv. \tag{3.3}
\]

Defining the piecewise gradient \( \nabla_h \) defined by \( (\nabla_h v)_K = \nabla v \) for each cell \( K \), and noticing that, when summing over \( K \in \Omega \), the interior cell interfaces are visited twice, giving rise to the jumps
\[
[v \mathbf{p}](x) := (v \mathbf{p}|_K \cdot \mathbf{n}_K)(x) + (v \mathbf{p}|_{K'} \cdot \mathbf{n}_{K'})(x)
\]
for \( x \in \partial K \cap \partial K' \), while for \( x \in \partial \Omega \cap \partial K \), we set 
\[
[v \mathbf{p}](x) := (v \mathbf{p}|_K \cdot \mathbf{n}_K)(x),
\]
summation over \( K \in \Omega_h \) yields the weak DG-formulation
\[
\int_\Omega -\mathbf{p} \cdot \nabla_h v + (c - \text{div} \mathbf{p}) u v + \int_{\partial \Omega_h} [v \mathbf{p}] \theta = \int_\Omega f v \, dx, \quad v \in V. \tag{3.4}
\]

For simplicity we wish to realize homogeneous inflow boundary conditions which explains the missing trace integral on the right-hand side. For (3.4) to be meaningful one has identify the (still infinite-dimensional) trial space \( U \) which is to host the pair \((u, \theta)\) of unknowns and the test space \( V \) which we now wish to be a broken space of the form (2.36). The first term on the left-hand side of (3.4) suggests taking \( V_K := H(p; K) \), i.e., we have
\[
H(p; \Omega_h) := \{v \in L_2(\Omega) | \quad p \cdot \nabla_h v \in L_2(\Omega)\} = \prod_{K \in \Omega_h} H(p; K) \tag{3.5}
\]
equipped with the norm
\[
\|v\|_{H(p; \Omega_h)} := \|v\|_{L_2(\Omega)}^2 + \|p \cdot \nabla_h v\|_{L_2(\Omega)}^2. \tag{3.6}
\]
which is the “broken” analog to the norm (2.4). While the solution component \( u \) only needs to belong to \( L_2(K) \) for each \( K \) and hence globally to \( L_2(\Omega) \), we need to identify a proper norm for the solution component \( \theta \) living on the skeleton \( \partial \Omega \). Recall that \( \theta \) really plays the role of a trace. The largest space of functions on \( \Omega \) which have traces in the weighted \( L_2 \)-spaces corresponding to the boundary integrals in (3.4) are in fact the spaces \( H(p; \Omega) \). This suggests taking
\[
H_{0, \Gamma_{-}}(p; \partial \Omega_h) := \{q|_{\partial \Omega_h} : q \in H_{0, \Gamma_{-}}(p; \Omega)\}
\]
endowed with the quotient norm
\[
\|\theta\|_{H_{0, \Gamma_{-}}(p; \Omega_h)} := \inf\{|q|_{H(p; \Omega)} : \quad \theta = q|_{\partial \Omega_h}, \quad q \in H_{0, \Gamma_{-}}(p; \Omega)\}.
\]

Leading to the (ultra-weak) formulation:
For \( f \in H(p; \Omega_h) \) find a \((u, \theta) \in U := \text{L}_2(\Omega) \times H_{0, \Gamma_{-}}(p; \partial \Omega_h)\) such that for all \( v \in V := H(p; \Omega_h) \) one has
\[
b_h((u, \theta), v) = f(v)
\]
with
\[
b_h((u, \theta), v) := \int_\Omega -\mathbf{p} \cdot \nabla_h v + (c - \text{div} \mathbf{p}) u v + \int_{\partial \Omega_h} [v \mathbf{p}] \theta. \tag{3.7}
\]
Note that, although being infinite-dimensional, the spaces $U, V$ depend on $\Omega_h$, which we suppress in the notation. Likewise the induced operator $B = B^{(b)}$ depends on the mesh $\Omega_h$.

It is shown in Broersen et al. [11] (Theorem 4.8) that the above (still infinite-dimensional) variational formulation for the given pair of trial and test spaces is uniformly well-conditioned with respect to hierarchies of shape regular meshes $\partial \Omega_h$, i.e., $\kappa_{U,V}(B^{(b)})$ remains (under mild conditions on the variable flow field $p$) uniformly bounded in $h$. In particular, the inf-sup constants remain safely bounded away from zero, i.e.,

$$b_h((u, \theta); v) \geq \|\theta\|_U \|v\|_V,$$

uniformly in $h$. Hence, residual error bounds of the form (2.21) are valid.

3.2. Well-conditioned Petrov-Galerkin schemes

It still remains to discretize (3.7), i.e., to choose finite dimensional trial and test spaces. A canonical choice would be to employ (globally discontinuous) piecewise polynomials subordinate to the partition $\Omega_h$ for the component $u_h$ while, due to the choice of $H_{0,\Gamma}(p; \partial \Omega_h)$, the $\theta$ component should be approximated by globally continuous piecewise polynomials on the skeleton. Since the norm for the skeleton component is actually somewhat stronger than for the $u$-component, the polynomial degree for $\theta$ should exceed the one for $u$. We denote by $P_m(K)$ the space of polynomials of (total) degree $m$ over $K$. Before specifying the finite dimensional trial space some comments on the corresponding local test-search spaces are in order. We think of the partitions $\Omega_h$ as members of a hierarchy of nested partitions where the subgrid $h$ is to be viewed as a loose reference to the meshsize which nevertheless may vary locally in such a way that shape regularity of the cells is preserved. In particular, local refinements subject to some fixed refinement rule (such as bisections or isotropic subdivisions) are permitted. Trial and test spaces will be actually built on pairs of meshes $\Omega_H, \Omega_h$ in the following sense. Given a cell $K'$ in a mesh $\Omega_H$, say, the local test-search space associated with $K'$ will be comprised of piecewise polynomials subordinate to a subgrid of (local) mesh size $h$ obtained by a fixed number of subdivisions of $K'$. The union of these subgrids in turn provide the global mesh $\Omega_h$ which means that each cell $K \in \Omega_h$ is the result of a fixed number of refinements of an ancestor $K' \in \Omega_H$. The trial functions will be piecewise polynomial on $\Omega_H, \partial \Omega_H$ while the test functions will be piecewise polynomials on the finer mesh $\Omega_h$. Thus, we use the finer mesh size $h$ to index the DG bilinear form. The finite-dimensional trial spaces $U^H$ are then of the form

$$U^H = \left( \prod_{K \in \Omega_H} P_m(K) \right) \times \left( \prod_{K \in \Omega_H} P_{m+1}(K) \right) \cap H(p; \Omega) \bigg|_{\partial \Omega_h}.$$

Note that the traces of the elements in $P_{m+1}(K)$ actually are evaluated on the skeleton of the subgrid $\partial \Omega_h$.

Although the trial functions are associated with the coarse mesh $\Omega_H$ the broken test-search space is associated with the subgrid $\Omega_h$, i.e., $V_K = H(p; K), K \in \Omega_h$. Next, we need to choose for each $V_K$ a finite-dimensional test-search space $V^h_K$. A proper choice for $V_K$ should ensure on the one hand, that uniform inf-sup stability is preserved, i.e., $T_h(U^h)$ is sufficiently close to $T(U^h)$, while on the other hand, the computational
complexity should remain acceptable. To prove that for \( U^H \) defined by (3.8),

\[
V^h_K = P_{m+2}(K), \quad \text{i.e.,} \quad V^h = \left( \prod_{K \in \Omega_h} P_{m+2}(K) \right)
\]

(3.9)
is indeed suitable, is much more involved, see [11]. Indeed it is shown there that under mild conditions on the reaction coefficient \( c \) and the (variable) convection field \( p = p(x) \), a uniformly bounded subgrid depth \( H/h \) suffices to guarantee uniform inf-sup stability, i.e.,

\[
\inf_{(w_h, \zeta_h) \in U^h} \sup_{v_h \in T_h(U^H)} \frac{b_h((w_h, \zeta_h); v_h)}{\| (w_h, \zeta_h) \|_V \| v_h \|_V} \geq \tilde{\gamma} > 0,
\]

holds for a positive \( \tilde{\gamma} \), uniformly in \( h \). Thus, since \( \#(\Omega_h) \leq C \#(\Omega_H) \), local test-search spaces of fixed uniformly bounded dimension suffice to ensure uniformly well-conditioned DPG-formulations. As argued at the end of Section 2.3 this is an essential precondition for keeping the computational work (ideally) proportional to the problem size. One should note though that in all numerical tests one observes that \( h = H \) suffices, i.e., the use of subgrids for the test-search spaces does not seem to be necessary in practice.

Since continuity (2.12) is inherited from the infinite-dimensional formulation the Petrov-Galerkin approximations given by

\[
b_h((u_h, \theta_h); v_h) = f(v_h), \quad v_h \in T_h(U^H), \quad (3.10)
\]
enjoy the best-approximation properties (2.35).

3.3. Implementation

Our implementation of (3.10) for the numerical solution of the PDE (2.1), is based on the recently developed DUNE-DPG module [13] that is built upon the finite element package DUNE [23]. Here we are content with a few remarks on the main ingredients. While

\[
(w^h, v^h)_{V_K} := \int_K w^h v^h + \int_K (p \cdot \nabla w^h)(p \cdot \nabla v^h)
\]

(3.11)
is the standard inner product on the local space \( V_K = H(p; K) \) which could be used in (2.41) for the computation of the test basis functions. However, the scalar product

\[
\langle v, z \rangle_{K,p} := \langle \partial_p v, \partial_p z \rangle_{L_2(K)} + \int_{\partial K} \frac{v(s) z(s)}{|n_K(s)|} r(s) ds,
\]

proposed in [11], gives rise to an equivalent norm on \( H(p; K) \) and shows a slightly better quantitative performance in applications and is crucial in the analysis of the local test-search spaces. Here \( r(s) \) is the distance of the point on \( \partial K \) with parameter \( s \) on the characteristic path to the corresponding exit point on the outflow boundary.

When the coefficients \( c, p \) depend on \( x \) one has to employ quadrature, or equivalently replace them cellwise by quantities that can be integrated exactly. Denoting by \( w \) a polynomial extension of the trace \( \theta \) into the cell, the following alternate representation for the local DG bilinear form \( b_K \) from (3.3)

\[
\int_K (\tilde{c} v - \text{div}(\tilde{p} v)) u + \tilde{d} u v + \int_{\partial K} p \cdot n_K v u d s, \quad K \in \Omega_h,
\]

should be used where \( \tilde{p}, \tilde{c}, \tilde{d} \approx p, c, \text{div} p, \text{div} p, \text{div} p \) respectively, see [11].
The following numerical experiments refer however to the simpler case of constant coefficients

\[ c = 0, \ p = (0.8, 0.6), \ f = 1. \]

We discretize the space \( U = L_2(\Omega) \times H_{0,\Gamma}(\mathbf{p}; \partial \Omega) \) by \( U^H \), defined in (3.8) for \( m = 2 \) which then fixes the test-search space according to (3.9). As basis functions for the two components \( \diamond \) and \( \bullet \) in (3.8) we employ local Lagrange polynomials of degree two and three, respectively, with respect to standard interpolation grids for the cells and their boundaries, respectively. According to (3.9), Lagrange elements of degree 4 are used for the test-search space \( V^H \) because they conveniently allow one to realize global continuity.

We solve the two-dimensional transport equation (2.1) on a unit square \( \Omega = (0, 1)^2 \) using a quadrilateral \( 20 \times 20 \) grid and therefore 5721 degrees of freedom. The inner product that induces the test norm (3.6) is given by (3.11) and is employed in (2.41) to calculate the near-optimal test functions. In Figure 1 three computed test functions are displayed that use fourth order polynomials.

Figure 2 shows an approximate solution for homogeneous inflow boundary conditions on \( \Gamma^- = \{ 0 \times [0, 1] \} \cup \{ [0, 1] \times 0 \} \). It increases along the flow direction \( \mathbf{p} \). Due to the nonsmoothness of \( \Gamma^- \), there is a kink in the solution starting at \( (0, 0) \) and propagating along \( \mathbf{p} \). Figure 3 shows how the \( L_2 \) error decreases when the mesh is refined. To calculate the \( L_2 \) error the exact solution was used. For a posteriori error bounds and non-constant convection fields we refer the reader to [11].

4. Application to the porous media problem

Coming back to the porous media problem introduced in [1], we apply the DPG approach to the nonlinear hyperbolic system, that consists of the continuity equation

\[ \nabla \cdot (\rho \ V) = 0 \] (4.1)

and the momentum equation

\[ \rho \phi^{-2} (V \cdot \nabla) V = -\nabla p - \frac{\mu}{K_D} V - \frac{\rho}{K_F} \|V\|_2 V. \] (4.2)

Here \( \phi \) denotes the porosity of the material, \( \rho \) the fluid density, \( V \) the Darcy velocity, \( \mu \) the dynamic viscosity of the fluid, \( K_D \) the permeability tensor of the medium and \( K_F \) the Forchheimer coefficient, which is also a tensor. The pressure \( p \) is determined by the equation of state for a thermally and calorically perfect gas, with the specific gas constant \( R \) given by

\[ p = \rho RT \]

with the fluid temperature of the coolant \( T \) also being an input parameter.

Since the system does not contain any diffusion and is governed by nonlinear convection we had previously employed artificial diffusion in order to stabilize a Galerkin scheme for its approximate solution. More precisely, we have employed a streamline-upwind Petrov-Galerkin stabilization which can, actually, be interpreted as a Petrov-Galerkin scheme. However, a proper choice of stabilization parameters is only known for very special convection-diffusion model problems and even then it cannot realize
the stability properties needed to ensure (2.35) and (2.21). Therefore, this approach is prone to either overmeasuring or still producing oscillations.

Therefore, we have prepared the ground for using the above DPG concepts for the numerical solution of the system (4.1), (4.2). Since the system is nonlinear and the DPG framework addresses linear problems we have to combine them with suitable linearizations of (4.1), (4.2) which, however, should be formulated already on the infinite-dimensional level. The resulting linear systems of transport equations can then be treated by a DPG method along the above lines.

To illustrate the principle we consider here only a Picard iteration as the simplest version to linearize the PDE and subsequently apply the DPG discretization. We start
with an initial guess $V_0$ and solve the linear transport problem

$$\nabla \cdot (\rho_1 V_0) = 0$$

(supplemented by suitable boundary conditions) for the scalar field $\rho_1$. The result is then used to solve the linear transport system

$$\rho_1 \varphi^{-2}(V_0 \cdot \nabla)V_1 = -\nabla(\rho_1 RT) - \frac{\mu}{K_D} V_1 - \frac{\rho_1}{K_F} \|V_0\|_2 V_1$$

for $V_1$. This process is to be repeated yielding a sequence of densities $\rho_i$ and velocities $V_i$ iteratively until a given error constant is reached. Recall that a posteriori residual based error bounds of the type (2.21) allow one, in principle, to monitor the accuracy. Alternatively, one could use a Newton iteration which also requires iteratively solving linear transport problems. Ideally this should be combined with nested iteration, i.e., the initial linearization is treated on a rather coarse mesh. The a-posteriori bounds tell us when the iteration can be terminated on that discretization level and the current approximation can be used as an initial guess on a refined mesh. In this sense the numerical scheme can be viewed as a perturbed iteration of an ideal iteration in function space where accuracy tolerances are dynamically updated so as to ensure overall convergence to
the exact solution. Ongoing work is concerned with analyzing the convergence of such
iterations and with working out the specific DPG ingredients for the above systems.

Regarding the last issue, given the velocity field \( V_{i-1} \), the approximate solution to
the \( i \)th iterate of (4.3) reads: find \( (\rho_i, \theta_i) \in L^2(\Omega) \times H_0(\Gamma_{i-1}; \partial \Omega_h) \) such that for all
\( v \in H(V_{i-1}; \Omega_h) \) one has

\[
b_h((\rho_i, \theta_i), v) = 0
\]

with

\[
b_h((\rho_i, \theta_i), v) \equiv \int_\Omega -\rho_i V_{i-1} \cdot \nabla h v + (c - \text{div} V_{i-1}) \rho_i v + \int_{\partial \Omega_h} [v V_{i-1}] \theta_i.
\]

Here \( R \) denotes the reservoir inflow boundary.

Since in the porous media application we have inhomogenous boundary conditions.
To deal with these boundary conditions we follow [11] (Remark 3.6):
let \( g \in H(V_{i-1}; \Omega_h) \) be an extension of \( \rho_R \) to \( \Gamma_{R(\text{reservoir})} \). Introducing \( \bar{\theta}_i := \theta_i - \bar{\rho}|\partial \Omega_h, \) the problem (4.5) takes
the form

\[
b_h((\rho_i, \bar{\theta}_i), v) = -\int_{\partial \Omega_h} [v V_{i-1}] g.
\]

Rearranging (4.4) and abbreviating parameters as \( a := -\nabla(\rho_iRT), B := \rho_i \phi^{-2} V_{i-1} \)
and \( c := \frac{K}{\mu D} + \frac{\rho_1}{K} \| V_{i-1} \|_2 \) we get the transport equation

\[
B \cdot \nabla V_i + c V_i = a.
\]

Following the lines of Section 3 and using again [11] (3.6) with \( G \in (H(V_{i-1}; \Omega_h))^2 \)
being an extension of \( V_y \) to \( \Gamma_{H(ot)G(ot)} \) we arrive at the following variational formulation:

For \( a \in H(V_{i-1}; \Omega_h)^{\prime} \) find \( (V_i, \sigma_i) \in (L^2(\Omega) \times H_0(\Gamma_{i-1}; \partial \Omega_h))^2 \) such that for all
\( W \in (H(V_{i-1}; \Omega_h))^2 \) one has

\[
b_h((V_i, \sigma_i), W) = -\int_{\partial \Omega_h} [W] V_{i-1} G
\]
with

\[
b_h((V_i, \sigma_i), W) \equiv \int_\Omega -V_i B \cdot \nabla h W + (c - \text{div} B) V_i \cdot W + \int_{\partial \Omega_h} [W] B \sigma_i.
\]
This is the central numerical problem to be solved in each iteration of the porous media problem. We make use of the DUNE-DPG module [13] that has been designed recently at the IGPM. Since the transport problem in the porous media model is non-linear, the present tool needs adaptations for iterative processes, which is work in progress. From the analytical point of view one faces two major tasks: first, one has to contrive an outer iteration in function space that can be shown to converge with a fixed error reduction per step. The Picard iteration shown above is only one of several options. Since the DPG scheme is equivalent to a minimum residual method in $V'$ one could also think of Gauss-Newton iterations in this metric. Second, for the arising linear problems one has to identify suitable finite-dimensional local test-search spaces that guarantee uniform inf-sup stability. Of course, it is again important that these local test-search spaces can be kept at a uniformly bounded dimension.

5. Conclusion and future work

The concept of the Discontinuous Petrov-Galerkin method has been introduced and explained in some detail for a linear transport equation. In particular, we have derived the corresponding variational formulation and the appropriate function spaces needed for a well-conditioned variational formulation. By using the recently developed DUNE module Dune-DPG [13] we have illustrated the method by explicitly computing optimal test functions as well as the solution of the transport problem. The main purpose of discussing first in some detail a simple problem is to illustrate the various steps needed in a DPG method and to prepare for subsequent applications to a nonlinear system. We have also indicated the tasks that arise in this context.

Furthermore we have presented a variational formulation for the transport part of the linearized porous media problem presented in [1]. Here we strongly rely on the stability result with the desired scaling properties in [11]. The nonlinearity of the porous media problem is the challenging part of the present problem. In [25], Demkowicz et al. present numerical results for the DPG method for nonlinear equations, namely, the (viscous) Burgers equation and the Navier-Stokes equations. However, there doesn’t seem to exist yet a complete theoretical underpinning for those approaches. So, there remains an enormous range of unresolved issues arising in connection with DPG methods for nonlinear problems.

To exemplify the principal strategy we have sketched a Picard iteration leading to a sequence of linear (still infinite-dimensional) problems which then can be treated by DPG concepts, preferably in connection with nested iteration. A DPG implementation of this formulation is the next step to obtain numerical results. To ensure that only local problems have to be solved, one has to show that appropriate broken test spaces give rise to uniformly inf-sup stable variational formulations. In parallel the convergence of the (outer) iteration needs to be analyzed. We note that the understanding of the involved linearized problems would carry, in principle, over to alternate versions of an outer iteration.

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